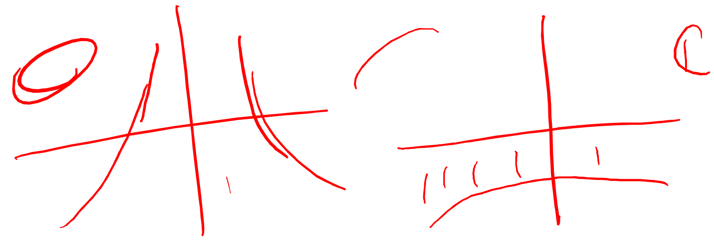
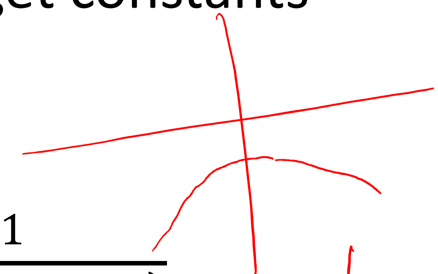
4.6 – Singularities and Resonance Free Regions

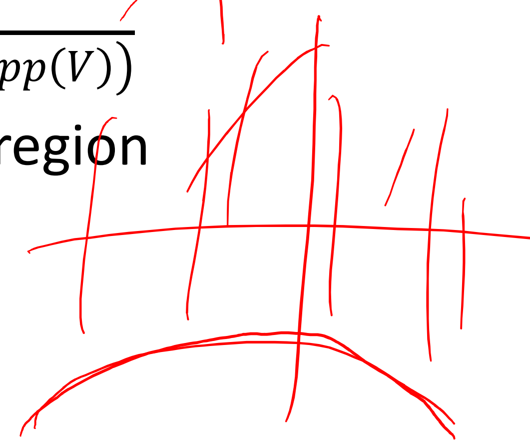
Izak

Recall: Theorem 3.10



- $V \in V^{\infty}_{comp}(\mathbb{R}^n, \mathbb{C}), n \geq 3, odd$. For any $\rho \in C_c^{\infty}(\mathbb{R}^n)$, get constants A, C, T depending on ρ with
- $\|\rho R_V \rho\|_{L^2 \to H^j} \leq C|\lambda|^{j-1} e^{T(Im\,\lambda)_-}\ j = 0,1,2$
- For $Im\,\lambda \geq -A - \delta \log(1 + |\lambda|), |\lambda| > C_0, \delta < \frac{1}{diam(supp(V))}$
- In particular, there are finitely many resonances in the region
- $\{\boldsymbol{\lambda}: \mathfrak{I}\boldsymbol{\lambda} \geq -\mathbf{A} - \boldsymbol{\delta}\, \boldsymbol{log}(\mathbf{1} + |\boldsymbol{\lambda}|)\}$

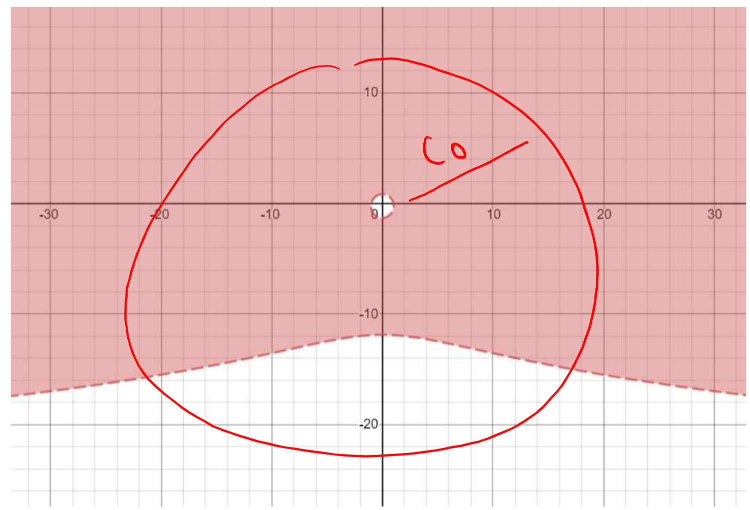




- **Now**: want to make this region arbitrarily large.

Theorem 4.41 (non-trapping estimates for smooth potentials)

- Suppose $V \in C_c^\infty(\mathbb{R}^n, \mathbb{R}), n \geq 3, odd,$ $R_V(\lambda) := \left(-\Delta + V - \lambda^2\right)^{-1}, \Im\lambda > 0.$
- For any $M > 0, \exists\, C_0$ such that $R_V(\lambda)\colon L^2_{comp} \to L^2_{loc}$ continues holomorphically to
- $\Omega_M := \{\lambda \in \mathbb{C} : \Im\lambda > -M \log|\lambda|, |\lambda| > C_0\}$
- And , for any $\chi \in C_c^\infty(\mathbb{R}^n)$ there are C_1, T with
- $\|\chi R_V(\lambda)\chi\|_{L^2 \to L^2} \leq C_1 |\lambda|^{-1} e^{T(\Im\lambda)_-}$



$|f^{(n)}| \leq C^n n^{an}$

Proof Outline

1. From spectral theorem, use wave propagator to rewrite resolvent
2. this expression is only valid for $\Im\lambda > 0$
3. Use cutoffs to make integrals converge
4. add correcting factors to get approximate resolvent
5. use Nuemann series to get final resolvent
6. establish bounds on norms to get result.

Proof – Wave Propagator

- Let $U_V(t) := \frac{\sin\left(t\sqrt{-\Delta+V}\right)}{\sqrt{-\Delta+V}} : L^2(\mathbb{R}^n) \to H^1(\mathbb{R}^n)$
- So $(P_V + \partial_t^2)U_V = 0, U_V(0) = 0, \partial_t U(0) = \mathbb{I}$

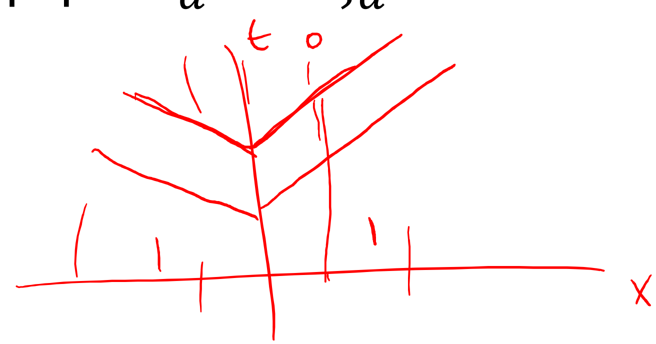
heaviside.

- Then $\overset{-\Delta+V}{R_V}(\lambda) := \int_0^\infty e^{i\lambda t}\, U_V(t)dt$ $\quad \mathcal{F}^*_{t\to\lambda}\left[U_V(t)H(t)\right]$
- This is true for $\Im\lambda > 0$, so we gotta use a bunch of cutoff functions.

Proof – Wave propagator cutoff

Supp $V \subset B(0,a)$
$\chi_a \equiv 1$ on supp V

- Let $U_V(t) := \frac{\sin\left(t\sqrt{-\Delta+V}\right)}{\sqrt{-\Delta+V}} : L^2(\mathbb{R}^n) \to H^1(\mathbb{R}^n)$
- So $(P_V + \partial_t^2)U_V = 0, U_V(0) = 0, \partial_t U(0) = \mathbb{I}$
- Let: $\widetilde{R_a}(\lambda)g = \int_0^\infty \zeta_a(x,t)e^{i\lambda t}U_V(t)[\chi_a g](x)dt : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$
- with $\chi_a \in C_c^\infty\big(B(0,a)\big), \chi_a V = V$
- $\zeta_a(x,t) \in C^\infty$ with $\zeta_a = 1$ for $t \leq |x| + T_a$ and $\zeta_a = 0$ for $t \geq |x| + T_a + 1$



Proof – Wave propagator cutoff

- Let: $\widetilde{R_a}(\lambda)g = \int_0^\infty \zeta_a(x,t)e^{i\lambda t}U_V(t)[\chi_a g](x)dt : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$

Proof – does $\widetilde{R_a}(\lambda)$ work?

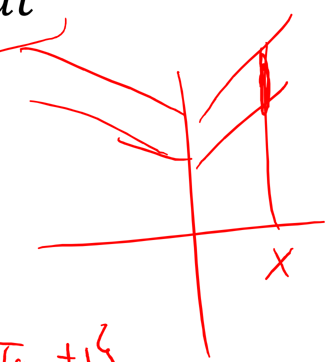
- $\widetilde{R_a}(\lambda)g = \int_0^\infty \zeta_a(x,t)e^{i\lambda t}U_V(t)[\chi_a g](x)dt : L^2 \to L^2$
- $(-\Delta + V - \lambda^2)\widetilde{R_a}(\lambda)g = \int_0^\infty P_V \zeta_a(x,t)e^{i\lambda t}U_V(t)[\chi_a g](x)dt + \int_0^\infty -\lambda^2\, \zeta_a(x,t)e^{i\lambda t}U_V(t)[\chi_a g](x)dt$

$P_V - \lambda^2$

- $\int_0^\infty -\lambda^2\, \zeta_a(x,t)e^{i\lambda t}U_V(t)[\chi_a g](x)dt = \int_0^\infty (\partial_t^2 e^{i\lambda t})\zeta_a U_V(t)[\chi_a g](x)dt$
- $= \left(\partial_t e^{i\lambda t}\right)\zeta_a U_V(t)[\chi_a g](x)\Big|_{t=0}^{t=\infty} - \left(e^{i\lambda t}\right)\partial_t\left(\zeta_a U_V(t)[\chi_a g](x)\right)\Big|_{t=0}^{t=\infty} + \int_0^\infty e^{i\lambda t}\partial_t^2\left(\zeta_a U_V(t)[\chi_a g](x)\right)dt$
- $= \chi_a g + \int_0^\infty e^{i\lambda t}\partial_t^2\left(\zeta_a U_V(t)[\chi_a g](x)\right)dt$

Proof – does $\widetilde{R_a}(\lambda)$ work?

- $\widetilde{R_a}(\lambda)g = \int_0^\infty \zeta_a(x,t)e^{i\lambda t}U_V(t)[\chi_a g](x)dt : L^2 \to L^2$
- Therefore:
- $(-\Delta + V - \lambda^2)\widetilde{R_a}(\lambda)g = \chi_a g + \int_0^\infty e^{i\lambda t}(P_V + \partial_t^2)\left(\zeta_a U_V(t)[\chi_a g](x)dt\right.$
- $\equiv \chi_a g + \int_{-\infty}^\infty e^{i\lambda t} F_a(t)[g](x)dt$
- where $F_a(t) = \left(P_V + \partial_t^2\right)(\zeta_a U_V(t)[\chi_a \cdot])(x): L^2 \to L^2$
    - will show integral converges and $F_a(t) \in C^\infty\left(\mathbb{R}; \mathcal{L}(L^2, L^2)\right)$



will turn out $\operatorname{supp} F_a(t)\, g(x) \in \{(t,x) : |x| + T_a \le t \le |x| + T_a + 1\}$

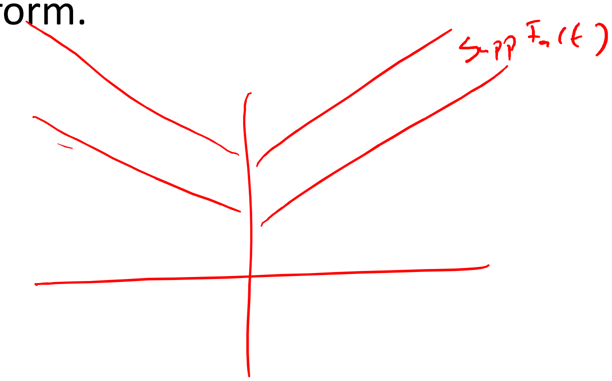
Proof – modify $\widetilde{R_a}$

- Now let $R_a^{\#}(\lambda)g \coloneqq \overbrace{\int_0^\infty e^{i\lambda t}\zeta_a(x,t)U_V(t)[\chi_a g](x)dt}^{\tilde{R}_a} + \int_{-\infty}^{\infty} e^{i\lambda t} W_a(t)[g](x)dt$
- $(-\Delta + V - \lambda^2)\int_{-\infty}^{\infty} e^{i\lambda t}\, W_a(t)[g](x)dt = \int_{-\infty}^{\infty} e^{i\lambda t}\, P_V\big(W_a(t)[g](x)\big)dt + \int_{-\infty}^{\infty}\big(\partial_t^2 e^{i\lambda t}\big)W_a(t)[g](x)dt$

  Hope $W_a(t)$ boundary terms = 0
- $= \int_{-\infty}^{\infty} e^{i\lambda t}\,\big(P_V + \partial_t^2\big)\big(W_a(t)[g](x)\big)dt$
- $= \int_{-\infty}^{\infty} e^{i\lambda t}\, V(x)W_a(t)[g](x)dx + \int_{-\infty}^{\infty} e^{i\lambda t}\,\big(-\Delta + \partial_t^2\big)W_a(t)[g](x)dt$
- Let's force $\big(-\Delta + \partial_t^2\big)W_a(t)[g](x) = -F_a(t)[\mathrm{g}]$ with compact support properties.

Proof – construct $W_a(t)$

- Let's force $(-\Delta + \partial_t^2)W_a(t)[g](x) = -F_a(t)[\mathrm{g}]$ with compact support properties.
- Want $(-\Delta + \partial_t^2)W_a(t) = -F_a(t), W_a(0) = \partial_t W_a(0) = 0$
- Just let $W_a(t)[g](x) = -\int_0^t U_0(t-s,x,y)\left[F_a(s)[g]\right](y)dyds$
    - turns out that $[W_a(t)g](x) \in C^\infty\left(\mathbb{R}_t, L^2(\mathbb{R}^n)\right)$ and $supp\left[W_a(t)[g]\right] \subset \{|x| + T_a \le t \le |x| + C_a\}$
    - get compact support in time, so we can take Fouier transform.



$\|\chi R(\lambda)\chi\|_{L^1\to L^2}$

Proof – finalize $R_a^\#$

$F_a(\cdot)$

$e^{i\lambda t}$

- $R_a^\#(\lambda)g := \int_0^\infty e^{i\lambda t}\zeta_a(x,t)U_V(t)[\chi_a g](x)dt + \int_{-\infty}^{\infty} W_a(t)[g](x)dt$
- We get:
- $(-\Delta + V - \lambda^2)(R_a^\#(\lambda)g) = \chi_a g + \int_{-\infty}^{\infty} e^{i\lambda t}\, F_a(t)[g](x)dt$
- $+ \int_{-\infty}^{\infty} e^{i\lambda t}\, V(x)W_a(t)[g](x)dx + \int_{-\infty}^{\infty} e^{i\lambda t}\, (-\Delta + \partial_t^2)W_a(t)[g](x)d$

$K_a(\lambda)(g) = \int_{-\infty}^{\infty} e$

- $= \chi_a g + \overset{\chi_a}{V}\underbrace{\int_{-\infty}^{\infty} e^{i\lambda t}\, W_a(t)[g]}_{K_a(\lambda)} = \chi_a g + \chi_a K_a(\lambda)[g] = \chi_a\big(I + K_a(\lambda)\big)[g]$
- If $K_a(\lambda)$ has small norm, then we get:
- $(-\Delta + V - \lambda^2)R_a^\#(\lambda)\big(I + K_a(\lambda)\big)^{-1}[g] = \chi_a g$
- So $R_a^\#(\lambda)\big(I + K_a(\lambda)\big)^{-1} = R(\lambda)\chi_a$

Proof – fill in gaps

- Gaps to fill:
  1. Prove Fourier transforms are defined and have desired mapping properties
  2. Prove norm bound on $K_a(\lambda)$ so $\left(I + K_a(\lambda)\right)^{-1}$ is defined
  3. Prove norm bound on $R_a^{\#}(\lambda)$ which give norm bounds on $R_V(\lambda)$

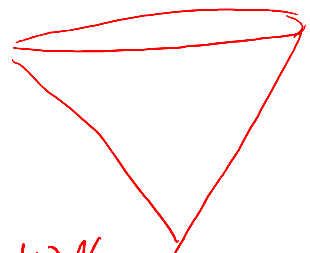
$$R_a^{\#}(\lambda)\left(I + K_a(\lambda)\right)^{-1} = R(\lambda)\chi_a$$

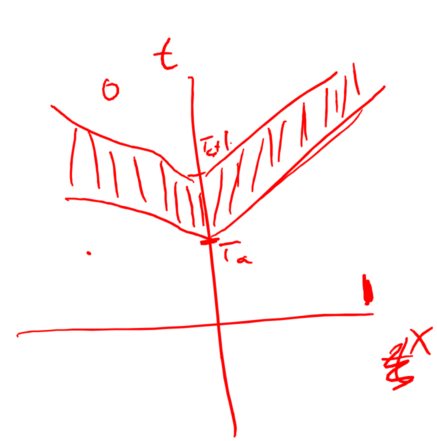
$$R_a^{\#}(\lambda)g := \underbrace{\int_0^\infty e^{i\lambda t}\zeta_a(x,t)U_V(t)[\chi_a g](x)dt}_{} + \underbrace{\int_{-\infty}^\infty e^{i\lambda t} W_a(t)[g](x)dt}_{}$$

Proof – Key part of proof!

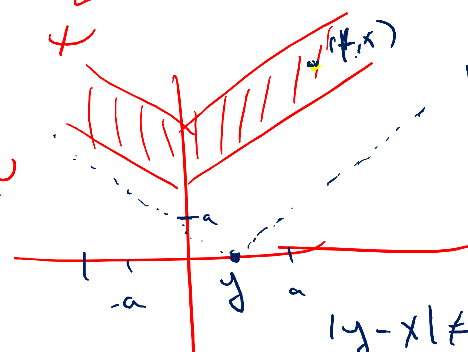
wave propagator $-\Delta + V + \partial_t^2$

- Why is $\int_0^\infty e^{i\lambda t} F_a(t) g = \int_0^\infty e^{i\lambda t}(P_V + \partial_t^2)\left(\zeta_a U_V(t)[\chi_a g](y)\right)dt$ smooth in time?
- **Key:** $supp U_V(t,\cdot) \subset \{(x,y) : |x-y| \le t\}$ and
- $\boldsymbol{singsupp\ U_V(t,\cdot) = \{(x,y) : |x-y| = |t|\}}$ **(theorem E.47)**
- $supp[(P_V + \partial_t^2), \zeta_a] \subset \{|x| + T_a \le t \le |x| + T_a + 1\} \equiv A$ ψ
- Now if $(t,x) \in A$ and $y \in supp\ \chi_a$, then $|x-y| \ne |t|$
- therefore $F_a(t) \in C^\infty(\mathbb{R}; \mathcal{L}(L^2, L^2))$ and compact support in time.





$F_a(t)\, g = \underbrace{[P_V + \partial_t^2, \zeta_a]}_{\psi} U_V(t) \chi_a g$



supp ψ

$|y-x| \ne t$

$\boxed{\psi\, U_V(t) \chi_a g}$

$\psi(t,x)\, U_V(t)\, (\chi_a(y)\, g(y))$

$\in C^\infty(\mathbb{R}_t, \mathbb{R}^{2n}_{(x,y)})$

Proof

bounded

- Why is $\int_{-\infty}^{\infty} W_a(t)\, g\, dt$ ~~defined~~?
- Recall $W_a(t)[g](x) = -\int_0^t \int_{-\infty}^{\infty} U_0(t-s,x,y)\big[F_a(s)[g]\big]\, dyds$
- by regularity and support of $F_a(t)$, get $W_a(t)g \in C^{\infty}(\mathbb{R}, L^2)$
- by a non-trivial argument, $supp\ W_a(t)g \subset \{|x| + T_a \le t \le |x| + C_a\}$

Norm Estimates

$u(t)$ comp supp distribution

$\Rightarrow$ $u \in C^\infty \Leftrightarrow \hat{u}(\lambda) = O(\lambda^{-\infty})$

$u \in C_c^\infty \Leftrightarrow \hat{u} = O(\lambda^{-\infty} e^{|\mathrm{Im}\,\lambda| c})$

- Bound $K_a(\lambda) = V \int_{-\infty}^{\infty} e^{i\lambda t}\, W_a(t)[g](x)dx$
- $[W_a(t)g](x) = -\int_0^t U_0(t-s,x,y)\,[F_a(s)g](y)dyds$
- $F_a(s) \in C^\infty\big(\mathbb{R}, \mathcal{L}(L^2, L^2)\big)$
- $\chi W_a(t) \in C_c^\infty\big((0,\infty), \mathcal{L}(L^2, L^2)\big)$ for any $\chi \in C_c^\infty(\mathbb{R}^n)$
- $\left\|\chi \int_{-\infty}^{\infty} e^{i\lambda t}\, W_a(t)[g]dt\right\|_{L^2} \le C_N \langle\lambda\rangle^{-N} e^{C_1(\Im\lambda)_-}$

$\partial_t($

~~$e^{i\lambda t} = e^{-\mathrm{Im}(\lambda)}$~~

Norm estimates

- By previous slide, get $\|K_a(\lambda)\|_{L^2\to L^2} \leq C_N\langle\lambda\rangle^{-N}e^{C_1(\Im\lambda)_-} < \frac{1}{2}$
  - for $\lambda \in \Omega_M \coloneqq \{\lambda \in \mathbb{C} : \Im\lambda > \ -M\log|\lambda|, |\lambda| > C_0\}$
- $R(\lambda)\chi_a = R_a^\#(\lambda)\big(I + K_a(\lambda)\big)^{-1}$
- $R_a^\#(\lambda)[g] = \int_0^\infty e^{i\lambda t}\zeta_a(x,t)U_V(t)[\chi_a g](x)dt + \int_{-\infty}^\infty W_a(t)[g](x)dt$
  - bound the second term by the same estimate from the previous slide
  - bound the first term by using $U_V(t), \partial_t U_V(t) = \mathcal{O}(\exp C|t|)_{L^2\to L^2}$

Non-Trapping

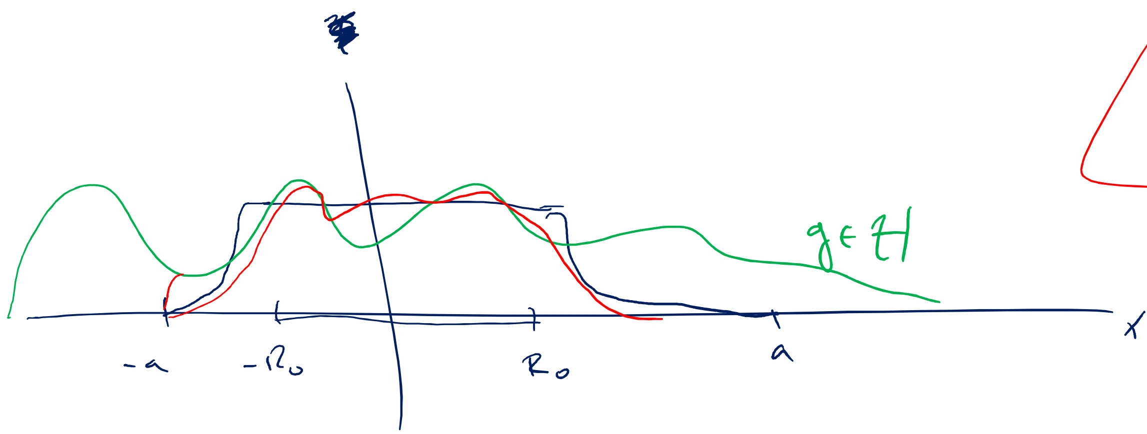
weird stuff happens

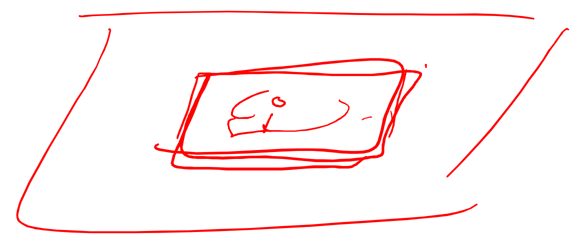
$B = B(0, R_0)$ $\quad$ $B^c = \mathbb{R}^n \setminus B(0, R_0)$

- **Definition (Non-trapping black box)**
- Given P is a black box Hamiltonian (self-adjoint operator on a Hilbert space)
  - $1_{B^c}\mathcal{D} \subset H^2(B^c)$
  - $1_{B^c}(Pu) = -\Delta(u|_{B^c})$
  - $v \in H^2(\mathbb{R}^n), v|_{B(0,R_0+\epsilon)} \equiv 0$ then $v \in \mathcal{D}$
  - $1_B(P(h) + i)^{-1}$ is compact
- P is nontrapping if $P \geq -C$ for some C and for all $a > R_0, \exists\, T_a$ such that for all $\chi \in C_c^\infty(B(0,a)), \chi|_{B(0,R_0+\epsilon)} \equiv 1$
- Then $\chi \frac{\sin(t\sqrt{P})}{\sqrt{P}} \chi\Big|_{t>T_a} \in C^\infty((T_a, \infty); \mathcal{L}(\mathcal{H}, \mathcal{D}))$

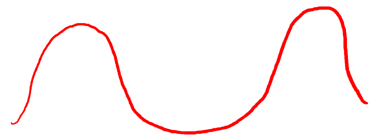
Non-Trapping

- P is nontrapping if $P \geq -C$ for some C and for all $a > R_0$, $\exists\, T_a$ such that for all $\chi \in C_c^\infty\big(B(0,a)\big)$, $\chi|_{B(0,R_0+\epsilon)} \equiv 1$
- Then $\chi \dfrac{\sin(t\sqrt{P})}{\underbrace{\sqrt{P}}_{U_V}} \chi \Big|_{t>T_a} \in C^\infty\big((T_a,\infty); \mathcal{L}(\mathcal{H},\mathcal{D})\big)$

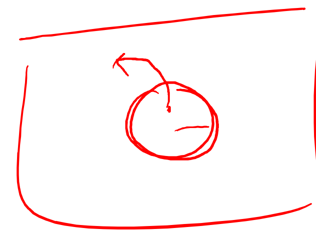




Non-Classical Trapping



- Given Riemannian metric g with $g^{ij} - \delta_{ij} \in C_c^\infty(B(0,R_0))$
- classical nontrapping: $\lim_{|t|\to\infty} \pi(\exp(tH_p(x,\xi))) = \infty \;\forall\; (x,\xi) \in T^*\mathbb{R}^n \setminus 0$
  - H_p Hamiltonian for $p(x,\xi) = \sum g^{ij}\xi_i\xi_j$, $\pi: T^*\mathbb{R}^n \to \mathbb{R}^n$ natural projection
- Implies for all $a > 0$, get T_a with $|x| < a, p(x,\xi) = 1, |t| > T_a$ then $|\pi((\exp tH_p)(x,\xi))| > a$
- by propagation of singularities: for all $\chi \in C_c^\infty(B(0,a))$ and $N > 0$ get: $\chi\left(\frac{\sin(t\sqrt{-\Delta_g})}{\sqrt{-\Delta_g}}\right)\chi \in C^\infty\left((T_a,\infty), \mathcal{L}(L^2(\mathbb{R}^n), H^N(\mathbb{R}^n))\right)$



$\Box_V(\delta(t-x\omega)) = V\delta(t-x\omega)$

$u(t,x,y) = \sum_{\pm} \int e^{i\varphi_\pm(t,x,\eta) - iy\eta} a_\pm(x,y,\eta)\,d\eta$

$U_V(t,x) = U_V^{\pm}(t,x,y) +$ smooth

$\Box E_1 = V\delta(t-x\omega)$

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Thm. 3.41

Eccl

solves $(\partial_t^2 - \Delta_g)u \in C^\infty$

$\Box_V = P_V + \partial_t^2$ $\quad \Box_0$

Solve E_1 $\quad E_1 = 0$ $\quad t \ll 0$

$\delta(t - x\omega) = H^{-1/2}$

$\Box E_1 = H^{1/2-}$ $\quad \Box E_2 = H^{3/2}$ ...

Propagation of singularities (idea) lecture 17

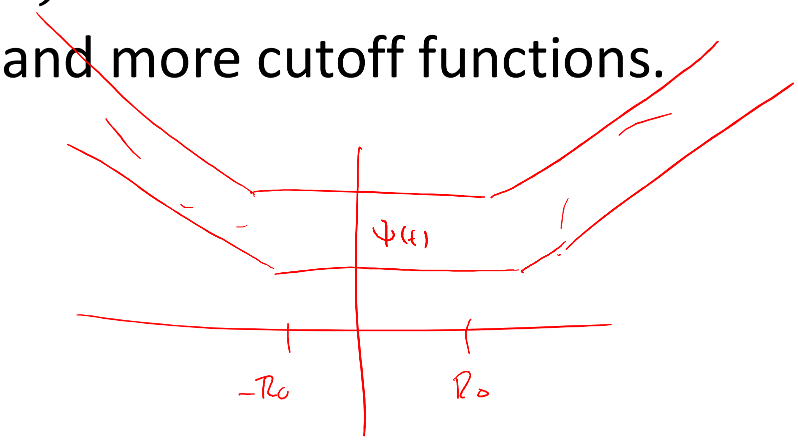
- singularities encoded in wavefront set $WF_h(u) \subset \overline{T^*\mathbb{R}^n}$
- $P = h^2(\partial_t^2 - \Delta) \to p = -\tau^2 + |\xi|^2 \to \exp\left(sH_p\right) = \phi_s$
- If $\phi_s(x_0, \xi_0) \notin WF_h(Pu)$ for $s \in (0, T)$ and $(x_0, \xi_0) \notin WF_h(u)$ then $\phi_T(x_0, \xi_0) \notin WF_h(u)$
- then if $Pu = \delta(t)\delta(x)$, we get that singularities of u are contained on the cone, propagating due to ϕ

Theorem 4.43 (Non-trapping estimates for black box Hamiltonians

- Given P non-trapping black box Hamiltonian with $R(\lambda)\colon \mathcal{H}_{comp} \to \mathcal{D}_{loc}$ the meromorphically continued reslovent
- Then for all M, $\exists\ C_0$ such that $R(\lambda)$ is holomorphic in $\Omega_M \coloneqq \{\lambda \in \mathbb{C}\colon \Im\lambda > -M \log|\lambda|, |\lambda| > C_0)$
- Also for all $\chi \in C_c^\infty(\mathbb{R}^n)$, $\chi = 1$ near $B(0, R_0)$ there exist C_1, T with
- $\|\chi R(\lambda)\chi\|_{\mathcal{H}\to\mathcal{H}} \leq C_1|\lambda|^{-1}e^{T(\Im\lambda)_-}$, $\lambda \in \Omega_M$

Proof – cutoff the propagator

- Let $supp\,\chi \subset B(0,a)$, $\chi_a \in C_c^\infty\big(B(0,a)\big)$ 1 on $supp\,\chi$, $\psi_a \in C^\infty(\mathbb{R} \times \mathbb{R}^n)$ with
- $supp\,\psi_a \cap \{\mathbb{R} \times B^c\} \subset \{(t,x) \colon |x| + T_a \le t \le |x| + T_a + 1\}$
- $\psi_a|_{\mathbb{R}\times B(0,R_0+\epsilon)} = \psi_a^0(t) \in C_c^\infty(T_a + R_0, T_a + R_0 + 1)$
- **Lemma:** $\psi_a U(t) \chi_a \in C^\infty\big(\mathbb{R}; \mathcal{L}(\mathcal{H}, \mathcal{D})\big)$
- proof: uses nontrapping assumption and more cutoff functions.



Proof – create $F_a(t)$

- Let $\zeta_a \in C^\infty$
- $\zeta_a(x,t)|_{|x|>R_0+2\epsilon} = \begin{cases} 1 & t \leq |x| + T_a \\ 0 & t \geq |x| + T_a + 1 \end{cases}$
- $\zeta_a(x,t)|_{|x|<R_0+2\epsilon} = \zeta_a^0(t = \begin{cases} 1 & t \leq R_0 + \epsilon + T_a \\ 0 & t \geq R_0 + \epsilon + T_a + 1 \end{cases}$
- **Claim**: $F_a(t) \coloneqq [(\partial_t^2 + P), \zeta_a]U(t)\psi_a \in C^\infty\left(\mathbb{R}; \mathcal{L}\left(\mathcal{H}, \mathcal{D}^{\frac{1}{2}}\right)\right)$
- this follows from the previous lemma

Proof - Create $W_a(t)$

- Let $\chi_b \in C_c^\infty\big(B(0,a)\big)$ with $\chi_b = 1$ near $B(0,R_0)$ and $\chi_a = 1$ on $supp\chi_b$, then construct $W_a(t)$ so that:
- $(\partial_t^2 + \Delta)W_a(t) = -(1-\chi_b)F_a(t), W_a(t) \equiv 0, t \geq 0$
- It will turn out that:
- $supp[W_a(t)g](x) \subset \{(x,t)\colon |x| + T_a \leq t \leq |x| + C_a\}$
- $[W_a(t)g](x) \in C^\infty\left(\mathbb{R}_t, \mathcal{D}^{\frac{1}{2}}\right)$

Proof – Approximation of Resolvent

- Let $\chi_c \in C_c^\infty\big(B(0,a)\big)$, 1 near $B(0,R_0)$, $\chi_b = 1$ on $supp\ \chi_c$
- Let $R_a^\#(\lambda) = \int_0^\infty e^{i\lambda t}\zeta_a U(t)\chi_a\, dt + \int_{-\infty}^\infty e^{i\lambda t}(1-\chi_c)W_a(t)\, dt$
- By computation $(P-\lambda^2)R_a^\#(\lambda) = \chi_a\big(I + K_a(\lambda)\big)$
- with $K_a(\lambda) = \int_{-\infty}^\infty e^{i\lambda t}\, \chi_b F_a(t) + [\Delta, \chi_c]W_a(t))\mathrm{dt}$
- Then $R(\lambda)\chi_a = R_a^\#(\lambda)\big(I + K_a(\lambda)\big)^{-1}$

Theorem 4.44 (resonance expansion for non-trapping black box Hamiltonians)

- Given P a non-trapping black box Hamiltonian, and $w(t)$ solves
- $(\partial_t^2 + P)w(t) = 0, w(0) = w_0 \in \mathcal{D}_{comp}^{\frac{1}{2}}, \partial_t w(0) = w_1 \in \mathcal{H}_{comp}$
- then for all $A > 0$,

$$w(t) = \sum_{\substack{\lambda_j \in Res(P) \\ \Im\lambda_j > -A}} \sum_{l=0}^{m_R(\lambda_j)-1} t^l e^{-i\lambda_j t} f_{j,l} + E_A(t)$$

- where the sum is finite, $\sum_{l=0}^{m_R(\lambda_j)-1} t^l e^{-i\lambda_j t} f_{j,l} + E_A(t) = Res_{\mu=\lambda_j}\left((iR(\mu)w_1 + \lambda R(\mu)w_0)e^{-i\lambda\mu}\right)$
- $(P - \lambda_j)^{l+1} f_{j,l} = 0$
- and there is a control on the error.